J. Fluid Mech. (1968), *vol.* 33, *part* 3, *pp.* 445–455
Printed in Great Britain

Finite amplitude convection with changing mean temperature. Part 1. Theory

By RUBY KRISHNAMURTI

Institute of Geophysics and Planetary Physics,
University of California, Los Angeles, California†

(Received 17 August 1967 and in revised form 26 December 1967)

When a horizontal layer of fluid is heated from below and cooled from above with the mean temperature and physical parameters of the fluid constant, the two-dimensional roll is known to be the stable solution near the critical Rayleigh number. In this study, with the mean temperature changing steadily at a rate η, the Rayleigh number and the velocity and temperature fields governed by the Boussinesq equations are expanded in two parameters: η, and the amplitude ϵ. Hexagons are shown to be the stable solution near the critical Rayleigh number. The direction of the motion depends upon the sign of η. A finite amplitude instability is possible with an associated hysteresis in the heat flux as the critical Rayleigh number is approached from below or from above.

1. Introduction

The instability that occurs in a static layer of fluid heated below and cooled above has long been a source of fascination (Bénard 1900; Rayleigh 1916; Pellew & Southwell 1940; Chandrasekhar 1961). At the point of instability, the possible flows form an infinitely degenerate set. The finite amplitude studies of Malkus & Veronis (1958) and of Schlüter, Lortz & Busse (1965) were addressed to the question, which of these would be restricted by the non-linearities, and which would be realized in experiment. Schlüter *et al.* (1965) have shown that, with constant material properties, all three-dimensional flows are unstable and that the two-dimensional roll is the only stable solution.

However, past observations of convection near the critical point has often shown a three-dimensional character and in fact, flows that are clearly two-dimensional are rare. In the work of Tippleskirch (1956), Palm (1960), Busse (1962), Segel & Stuart (1962), and others, the selection of hexagons is attributed to the non-Boussinesq effect of variation of physical parameters, such as the viscosity, with temperature.

It may be noted that with a horizontally infinite geometry, rolls are symmetric in the sense that the sign of the vertical motion can always be altered by a horizontal translation of the reference axes. With hexagons, however, the direction of flow is an added degree of freedom to be determined. It would seem that the direction of flow, and perhaps a preference for hexagons, would depend upon

† Present address: Institute of Geophysical Fluid Dynamics, Florida State University, Tallahassee, Florida, 32306.

vertical asymmetries. In studies such as Palm's (1960) the variation of viscosity with temperature introduces a vertical asymmetry.

Another form of asymmetry which can, and often does, occur is in the heating. Convection can be obtained by only heating from below, or by only cooling from above, rather than symmetrically heating and cooling below and above. This is the case with some of the convection studies reported in the past. In particular, if the top of the layer is open to the air, or is covered by a poorly conducting transparent boundary for the sake of viewing the plan form of the cells, then the heat supplied from below is not carried away and the mean temperature must change. Also, in natural occurrence it would rarely happen that convection sets in with the mean temperature strictly fixed. If the mean temperature of the fluid is increasing steadily with time, the conduction temperature profile appears as in figure 1 (*a*). Defining η to be a rate of change of mean temperature, η is positive in this case. If the mean temperature is decreasing, η negative, the conduction temperature profile is as shown in figure 1 (*b*). It is this asymmetry and the associated removal of the degeneracy within the framework of the Boussinesq equations that is investigated here.

2. Governing equations

A layer of fluid of depth d and of infinite horizontal extent is bounded above and below by perfect conductors. An adverse temperature difference ΔT is maintained across the layer.

From the statements of conservation of mass, momentum and energy, the Boussinesq equations are derived as the lowest order in an expansion in two parameters, one proportional to the depth of the layer, the other to the temperature difference across it (Spiegel & Veronis 1960; Mihaljan 1962; Malkus 1964). They are written

$$(\partial_t - \kappa\nabla^2)T = -u_j\partial_j T, \tag{1}$$

$$(\partial_t - \nu\nabla^2)u_i = -\partial_i\pi - u_j\partial_j u_i + g\alpha(T-T_0)\lambda_i, \tag{2}$$

$$\partial_j u_j = 0, \tag{3}$$

where u_j is the jth component of velocity, $j = 1, 2, 3$, T is the temperature, π an effective pressure, κ the thermometric conductivity, ν kinematic viscosity, α the thermal expansion coefficient,

$$\partial_j \equiv \frac{\partial}{\partial x_j}, \quad \partial_t \equiv \frac{\partial}{\partial t}, \quad \lambda_i = (0,0,1).$$

The equation of state

$$\rho = \rho_0[1-\alpha(T-T_0)],$$

and the summation convention over repeated indices have been used. In this set of non-linear coupled equations for the velocity and temperature fields, the physical parameters ν, κ, α are constants to this lowest order. The density appears as a constant everywhere except in the buoyancy term, and the equation of mass conservation is, to this order, the statement of incompressibility. Variation of the physical parameters is a higher order effect.

The boundary conditions to be satisfied at the upper and lower surfaces are:

$w = 0, \quad \partial^2_{zz} w = 0$ at a stress-free boundary,

$u_i = 0, \quad \partial_z w = 0$ at a rigid (no slip) boundary,

$T = T_s$ (defined below) at perfectly conducting boundaries,

where z is the vertical co-ordinate, w the z-component of the velocity.

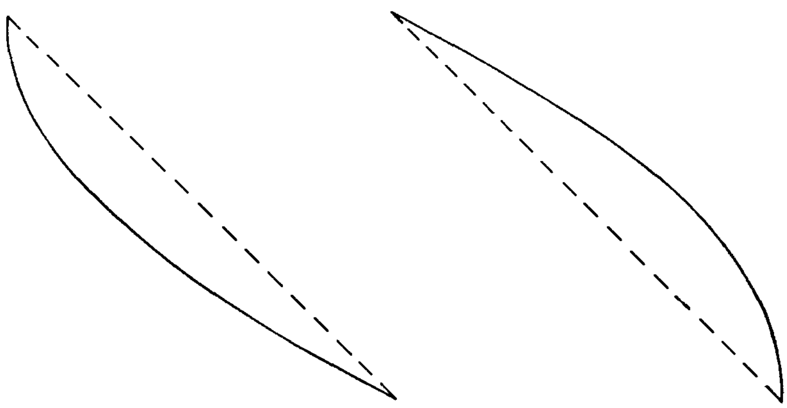
In the conduction state, equation (1) for the static temperature T_s becomes

$$(\partial_t - \kappa \partial^2_{zz}) T_s = 0$$

to be solved for the following boundary conditions:

$$T_s = \tfrac{1}{2}\Delta T + \eta t \quad \text{at} \quad z = -\tfrac{1}{2}d,$$

$$T_s = -\tfrac{1}{2}\Delta T + \eta t \quad \text{at} \quad z = \tfrac{1}{2}d.$$



(a) $\eta > 0$ (b) $\eta < 0$

FIGURE 1. Conduction temperature profiles.

We seek a steady solution of the form

$$T_s - T_r = -\frac{\Delta T}{d} z + \frac{\eta}{2\kappa}(z^2 - \tfrac{1}{4}d^2),$$

where T_r is a reference static temperature equal to ηt. This represents the case in which all points in the fluid are changing in temperature at the same rate as the boundaries, and the shape of the temperature profile is independent of time. Non-dimensionalizing by using d for length scale, d^2/κ for time scale, and ΔT for the static temperature, we obtain

$$T'_s - T'_r = -z' + \tfrac{1}{2}\eta'[z'^2 - \tfrac{1}{4}], \tag{4}$$

where

$$\eta' \equiv \frac{d^2}{\kappa \Delta T} \frac{\partial T_s}{\partial t}.$$

The total temperature is now written as

$$T = T_s + \theta.$$

The equation governing θ is then

$$(\partial_t - \kappa\nabla^2)\theta = -u_z\partial_z T_s - u_j\partial_j\theta$$
$$= \frac{\Delta T}{d}u_z - \frac{\eta}{\kappa}zu_z - u_j\partial_j\theta.$$

Non-dimensionalizing, using the scales

$$u = u'\kappa/d, \quad \theta = \theta'\frac{\kappa\nu}{g\alpha d^3},$$

we obtain the equations governing stationary solutions, neglecting primes:

$$\left.\begin{aligned} \nabla^2 u_i + \theta\lambda_i - \partial_i\pi &= \frac{1}{p_r}u_j\partial_j u_i, \\ \nabla^2\theta + Ru_z &= R\eta z u_z + u_j\partial_j\theta, \end{aligned}\right\} \tag{5}$$

where p_r = Prandtl number = ν/κ, and R = Rayleigh number = $(g\alpha/\kappa\nu)\Delta T d^3$.

Following the method used by Schlüter *et al.* (1965) to test the stability of the stationary solutions of (5), they will be perturbed by disturbances $(\tilde{u}_i, \tilde{\theta})$ of infinitesimal amplitude. The equations governing these perturbations, having growth rate σ, are

$$\left.\begin{aligned} \nabla^2\tilde{u}_i + \tilde{\theta}\lambda_i - \partial_i\tilde{\pi} - \frac{1}{p_r}\sigma\tilde{u}_i &= \frac{1}{p_r}(u_j\partial_j\tilde{u}_i + \tilde{u}_j\partial_j u_i), \\ \nabla^2\tilde{\theta} + R\tilde{u}_z - \sigma\tilde{\theta} &= R\eta z\tilde{u}_z + \tilde{u}_j\partial_j\theta + u_j\partial_j\tilde{\theta}. \end{aligned}\right\} \tag{6}$$

3. Method of solution

The stationary fields and the Rayleigh number are expanded as follows, using a double expansion in the parameters ϵ, η:

$$\left.\begin{aligned} u_i &= \sum_{n=1,\,m=0}^{\infty} \epsilon^n\eta^m u_i^{(n,m)}, \\ \theta &= \sum_{n=1,\,m=0}^{\infty} \epsilon^n\eta^m\theta^{(n,m)}, \\ R &= \sum_{n=0,\,m=0}^{\infty} \epsilon^n\eta^m R^{(n,m)}. \end{aligned}\right\} \tag{7}$$

We seek solutions of the non-linear equations (5) for small but finite amplitude. The problem of ordering the non-linearities by an expansion in powers of an amplitude ϵ was formulated by Malkus & Veronis (1958); the convergence of the expansion has been proven by Lortz (1961). Thus η appears as a given small parameter in equations having known solutions when η is zero. Substituting the expansions (7) into the governing equations (5) and using ϵ and η as ordering parameters we obtain an infinite set of linear inhomogeneous differential equations.

When the fields are expanded in powers of ϵ and η, R must also be so expanded, the $R^{(n,m)}$ being determined at each order in such a way as to satisfy the solubility conditions on the inhomogeneous equations.

Although the problem of testing the stability of the solutions of (5) would seem very difficult, it is noted that the perturbation equations (6) contain ϵ and η as given small parameters. Furthermore, when ϵ and η approach zero the equations (6) reduce to the linear stability problem which has known solutions. Thus it is natural to expand the perturbation fields and the growth rate σ as follows:

$$\left.\begin{aligned} \tilde{u}_i &= \sum_{n=1,m=0}^{\infty} \epsilon^n \eta^m \tilde{u}_i^{(n,m)}, \\ \tilde{\theta} &= \sum_{n=1,m=0}^{\infty} \epsilon^n \eta^m \tilde{\theta}^{(n,m)}, \\ \sigma &= \sum_{n=0,m=0}^{\infty} \epsilon^n \eta^m \sigma^{(n,m)}. \end{aligned}\right\} \tag{8}$$

The velocity and temperature fields are required to satisfy the boundary conditions at each order.

To order ϵ^1, η^0, equations (5) and (6) give the linear stability problem with well-known minimum eigenvalue, R_c, and separated eigenfunctions of the form

$$u_i^{(1,0)} = \delta_i v^{(1,0)}, \quad \text{where} \quad \delta_i = \begin{bmatrix} \partial x & \partial z \\ \partial y & \partial z \\ -\partial^2 x & -\partial^2 y \end{bmatrix},$$

$$v^{(1,0)} = \sum_{=-N}^{N} c_\rho w_\rho f(z), \quad w_\rho = \exp\{i\mathbf{k}_\rho . \mathbf{r}\}. \tag{9}$$

Here $\mathbf{r}$ is a horizontal position vector, $\mathbf{k}_\rho$ a horizontal wave vector.

The stationary equations to order ϵ^1, η^1 are:

$$\left.\begin{aligned} &\nabla^2 u_i^{(1,1)} + \theta^{(1,1)}\lambda_i - \partial_i \pi^{(1,1)} = 0, \\ &\nabla^2 \theta^{(1,1)} + R^{(0,0)} u_z^{(1,1)} = -R^{(0,1)} u_z^{(1,0)} + R^{(0,0)} z u_z^{(1,0)}. \end{aligned}\right\} \tag{10}$$

Multiplying the first part of (10) by $R^{(0,0)} u_i'^{(1,0)}$, the second by $\theta'^{(1,0)}$, and adding and integrating over the volume of the fluid, the solubility condition

$$R^{(0,1)} \langle \theta'^{(1,0)} u_z^{(1,0)} \rangle = R^{(0,0)} \langle \theta'^{(1,0)} z u_z^{(1,0)} \rangle$$

is obtained where $\theta'^{(1,0)}$, $u_i'^{(1,0)}$ are arbitrary solutions of order (1, 0). The brackets indicate an integration over the entire fluid, divided by the volume of fluid, in the limit of horizontal extent tending to infinity.

Since $\theta'^{(1,0)} u_z^{(1,0)}$ is an even function of z, the result is that $R^{(0,1)} = 0$. To order ϵ^1, η^2 the stationary equations are

$$\left.\begin{aligned} &\nabla^2 u_i^{(1,2)} + \theta^{(1,2)}\lambda_i - \partial_i \pi^{(1,2)} = 0, \\ &\nabla^2 \theta^{(1,2)} + R^{(0,0)} u_z^{(1,2)} = -R^{(0,2)} u_z^{(1,0)} + R^{(0,0)} z u_z^{(1,1)}. \end{aligned}\right\} \tag{11}$$

Multiplying the first by $R^{(0,0)} u_i'^{(1,0)}$, the second by $\theta'^{(1,0)}$, and adding and integrating over the volume of the fluid gives

$$R^{(0,2)} \langle \theta'^{(1,0)} u_z^{(1,0)} \rangle = R^{(0,0)} \langle \theta'^{(1,0)} z u_z^{(1,1)} \rangle.$$

For two free boundaries, $R^{(0,2)} = -1{\cdot}96$. For two rigid boundaries the critical Rayleigh number can be computed, including all η orders, from the critical

Taylor numbers for Couette flow between two cylinders rotating in the same direction. The adjoint relationship of the two problems has been pointed out by Debler (1966), and critical Rayleigh numbers computed by Debler (1966), Sparrow, Goldstein & Jonsson (1964).

To order ϵ^2, η^1, the stationary equations are

$$\left.\begin{aligned}\nabla^2 u_i^{(2,1)} + \theta^{(2,1)}\lambda_i - \partial_i \pi^{(2,1)} &= \frac{1}{p_r} u_j^{(1,0)}\partial_j u_i^{(1,1)} + \frac{1}{p_r} u_j^{(1,1)}\partial_j u_i^{(1,0)},\\ \nabla^2\theta^{(2,1)} + R^{(0,0)}u_z^{(2,1)} &= -R^{(1,1)}u_z^{(1,0)} + R^{(0,0)}zu_z^{(2,0)} + u_j^{(1,0)}\partial_j\theta^{(1,1)} + u_j^{(1,1)}\partial_j\theta^{(1,0)}.\end{aligned}\right\} \quad (12)$$

Multiplying the first by $R^{(0,0)}u_i'^{(1,0)}$, the second by $\theta'^{(1,0)}$, adding and integrating over the volume of the fluid, we obtain from the solubility condition,

$$\begin{aligned}R^{(1,1)}\langle\theta'^{(1,0)}u_z^{(1,0)}\rangle = \frac{R^{(0,0)}}{p_r}(\langle u_i'^{(1,0)}u_j^{(1,0)}\partial_j u_i^{(1,1)} + u_i'^{(1,0)}u_j^{(1,1)}\partial_j u_i^{(1,0)}\rangle)\\ + R^{(0,0)}\langle\theta'^{(1,0)}zu_z^{(2,0)}\rangle + \langle\theta'^{(1,0)}u_j^{(1,0)}\partial_j\theta^{(1,1)} + \theta'^{(1,0)}u_j^{(1,1)}\partial_j\theta^{(1,0)}\rangle. \quad (13)\end{aligned}$$

The right-hand side of (13) can be transformed in such a way that $R^{(1,1)}$ can be computed without the solution $\theta^{(1,1)}$, $u_j^{(1,1)}$, giving

$$R^{(1,1)}\langle\theta^{(1,0)}u_z^{(1,0)}\rangle = R^{(0,0)}\{\langle\theta^{(1,0)}zu_z^{(2,0)}\rangle - \langle\theta^{(2,0)}zu_z^{(1,0)}\rangle\}. \quad (14)$$

The right-hand side of (13) may be written as $H(\omega', \omega, \omega)$, as triple products of $(1, 0)$ solutions. Allowing the arbitrary ω' to range through all solutions, the integral H will be non-zero if, and only if, k_ρ, k_κ, k_λ form an equilateral triangle. $R^{(1,1)}$ vanishes unless a solution contains two k-vectors $\frac{2}{3}\pi$ apart. If it does contain two such k-vectors, then (12) is satisfied with $R^{(1,1)}$ non-zero only if the solution contains another $\mathbf{k}_\mu = \mathbf{k}_\lambda \pm \mathbf{k}_\kappa$, with equal coefficients c_κ, c_λ, c_μ:

$$\omega = \sum_{\kappa=-3}^{3} c_\kappa \exp\{i\mathbf{k}_\kappa.\mathbf{r}\}, \quad c_\kappa c_\kappa^* = \tfrac{1}{6}, \quad \sum_{\kappa=1}^{3}\mathbf{k}_\kappa = 0.$$

The results of the calculation are given in table 1.

The results to order ϵ^1, η^ν clearly give the critical Rayleigh number

$$R_c = R^{(0,0)} + \eta^2 R^{(0,2)} + \ldots$$

For two rigid boundaries, and $\eta = 8$ for example (Krishnamurti 1967)

$$R_c = 1537{\cdot}5.$$

This is a change of 10 % from the critical number when $\eta = 0$, and should be a measurable difference.

To first order in ϵ, second order in η, we have

$$\epsilon = \frac{R - R_c}{\eta R^{(1,1)}}. \quad (15)$$

The result that $R^{(1,1)}$ is non-zero leads us to conclude that a finite amplitude instability is possible, at a Rayleigh number less than the critical value R_c predicted by the linear theory. Had $R^{(1,1)}$ been equal to zero, it would be found that

$$\epsilon^2 = \frac{R - R_c}{R^{(2,0)}},$$

and since $R^{(2,0)}$ is positive, a real amplitude ϵ would be impossible at $R < R_c$. In other words, the linear stability theory would have correctly predicted the lowest Rayleigh number at which the fluid was unstable. Of course it is not necessary that an infinitesimal amplitude disturbance be the first to grow. We know that instability is often initiated by the nonlinear effects of small but finite disturbances. The finite amplitude study constitutes a critique of linear stability theory in that it allows us to state under what conditions the linear stability theory is meaningful. Here we find that $R^{(1,1)}$ is non-zero and that motions are possible below R_c. These must be finite amplitude disturbances since infinitesimal amplitude disturbances are known to decay below R_c. Their amplitude as given by (15) is seen to be opposite in sign to that of η for $R < R_c$.

Two free boundaries		
$R^{(0,0)} = 657$	$R^{(1,0)} = 0$	$R^{(2,0)} = \frac{9}{4}\pi^2\left[\frac{1}{2}+0{\cdot}189+\frac{0{\cdot}0457}{p_r}+\frac{0{\cdot}0709}{p_r^2}\right]$ for hexagons
$a^2 = \frac{1}{2}\pi^2$		
$R^{(0,1)} = 0$	$R^{(1,1)} = 5{\cdot}920+\frac{1{\cdot}1838}{p_r}$ for hexagons; $R^{(1,1)} = 0$ for all other forms	
$R^{(0,2)} = -1{\cdot}96$		
Two rigid boundaries		
$R^{(0,0)} = 1707{\cdot}762$; $a = 3{\cdot}117$	$R^{(1,0)} = 0$	$R^{(2,0)} = 27{\cdot}50+\frac{1{\cdot}526}{p_r}+\frac{2{\cdot}048}{p_r^2}$ for hexagons
$R^{(0,1)} = 0$	$R^{(1,1)} = 19{\cdot}94+\frac{8{\cdot}220}{p_r}$ for hexagons; $R^{(1,1)} = 0$ for all other forms	

TABLE 1. Results of the calculations.

From these results we may expect to observe the following: (i) a new critical Rayleigh number dependent upon η, (ii) hexagonal flows near the critical Rayleigh number with the direction of flow determined by the sign of η (the stability range of hexagonal flows is determined in §4 below), (iii) a hysteresis in the heat flux as the critical Rayleigh number is approached from below and from above. The horizontally averaged heat equation has a first integral H given in dimensionless form by

$$H' = R+\langle w'\theta'\rangle.$$

Solving (7) for ϵ gives

$$\epsilon = -\frac{\eta R^{(1,1)}}{2R^{(2,0)}} \pm \left[\left(\frac{\eta R^{(1,1)}}{2R^{(2,0)}}\right)^2 + \frac{R-R_c}{R^{(2,0)}}\right]^{\frac{1}{2}}. \tag{16}$$

The stability calculation of §4 below shows that $\epsilon\eta R^{(1,1)}$ must be negative for the stability of hexagonal flows. Calculation has shown that $R^{(2,0)}$ is positive. From this we can deduce the heat flux curve as shown in figure 2. The minimum of this curve is found at $(\partial H/\partial R)^{-1} = 0$ or equivalently $(\partial \epsilon/\partial R)^{-1} = 0$. Differentiating (16) gives us the Rayleigh number corresponding to the minimum as

$$R_{\min} - R_c = -(\eta R^{(1,1)})^2/4R^{(2,0)}. \tag{17}$$

At $R = R_{\min}$, we note that the corresponding amplitude is

$$\epsilon_{\min} = -\eta R^{(1,1)}/2R^{(2,0)}.$$

This is identical to ϵ_A, the minimum amplitude for the stability of hexagons, computed in §4 below.

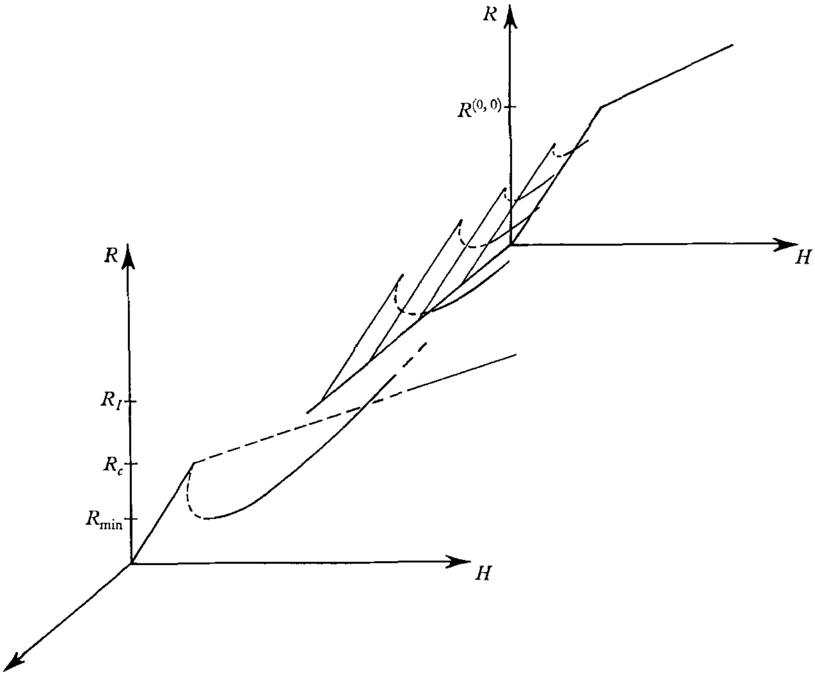


FIGURE 2. Heat flux H as a function of Rayleigh number R and rate of change of mean temperature η.

Another point of interest is the calculation of the Rayleigh number at which the heat flux curves for rolls intersects that for hexagons. Since $R^{(1,1)}$ is zero for rolls, we have that

$$\epsilon^2 = \frac{R - R_c}{R^{(2,0)}}.$$

The heat flux for roll cells near R_c is

$$H_{\text{rolls}} = R + N^*(\eta)\frac{R - R_c}{R^{(2,0)}},$$

while that for hexagonal cells is, from (16),

$$H_{\text{hex}} = R + N^*(\eta)\left\{2\left(\frac{\eta R^{(1,1)}}{2R^{(2,0)}}\right)^2 + \frac{R - R_c}{R^{(2,0)}} + \frac{\eta R^{(1,1)}}{2R^{(2,0)}}\left[\left(\frac{\eta R^{(1,1)}}{2R^{(2,0)}}\right)^2 + \frac{R - R_c}{R^{(2,0)}}\right]^{\frac{1}{2}}\right\},$$

where $$N^*(\eta) = \langle w^{(1,0)}\theta^{(1,0)}\rangle + \eta\langle w^{(1,1)}\theta^{(1,0)} + w^{(1,0)}\theta^{(1,1)}\rangle.$$

The two curves intersect at $R = R_I$ where

$$R_I = R_c + \frac{3(\eta R^{(1,1)})^2}{4R^{(2,0)}}. \tag{18}$$

The amplitude at $R = R_I$ is ϵ_I given by

$$|\epsilon_I| = \left|3\frac{\eta R^{(1,1)}}{2R^{(2,0)}}\right| \simeq 1{\cdot}5|\eta|.$$

4. The stability range of hexagonal flows

To be realized in experiment, the steady finite amplitude solutions must be stable at least to disturbances of infinitesimal amplitude. The stability of the finite amplitude flows is investigated in the manner of Schlüter *et al.* (1965) who have shown that $\sigma^{(1,0)}$ vanishes for all solutions and that $\sigma^{(2,0)}$ is positive for all except the roll solution.

When the mean temperature of the fluid layer is changing at a rate of η, there is the possibility of the term $\epsilon\eta\sigma^{(1,1)}$ compensating for a positive $\epsilon^2\sigma^{(2,0)}$.

The non-stationary equations to order ϵ^2, η^1 are:

$$\nabla^2\tilde{u}_i^{(2,1)} + \tilde{\theta}^{(2,1)}\lambda_i - \partial_i\tilde{\pi}^{(2,1)} = \frac{1}{p_r}\sigma^{(1,1)}\tilde{u}_i^{(1,0)} + \frac{1}{p_r}[\tilde{u}_j^{(1,1)}\partial_j u_i^{(1,0)} + u_j^{(1,1)}\partial_j\tilde{u}_i^{(1,0)} + \tilde{u}_j^{(1,0)}\partial_j u_i^{(1,1)} + u_j^{(1,0)}\partial_j\tilde{u}_i^{(1,1)}],$$

$$\nabla^2\tilde{\theta}^{(2,1)} + R^{(0,0)}\tilde{u}_z^{(2,1)} = \sigma^{(1,1)}\tilde{\theta}^{(1,0)} + R^{(0,0)}z\tilde{u}_z^{(2,0)} - R^{(1,1)}\tilde{u}_z^{(1,0)} + \tilde{u}_j^{(1,1)}\partial_j\theta^{(1,0)} + \tilde{u}_j^{(1,0)}\partial_j\theta^{(1,1)} + u_j^{(1,1)}\partial_j\tilde{\theta}^{(1,0)} + u_j^{(1,0)}\partial_j\tilde{\theta}^{(1,1)}.$$

The solubility condition gives

$$-\sigma^{(1,1)}\langle M(w'\tilde{w})\rangle + R^{(1,1)}\langle N_0(w'\tilde{w})\rangle = \langle\tilde{H}(w'\tilde{w}w)\rangle, \tag{19}$$

where
$$\tilde{H}(w',\tilde{w},w) = H(w',\tilde{w},w) + H(w',w,\tilde{w}), \tag{20}$$

where H was defined for the stationary problem and where M is a positive definite quantity.

Letting w' run through the orthogonal set

$$w' = \exp\{i\mathbf{k}_\rho.\mathbf{r}\}, \quad (\rho = -N\ldots -1, 1\ldots N),$$

we obtain from (19) a system of equations for the coefficients $\tilde{c}$ of $\tilde{w}$. The solubility condition is then satisfied if the secular determinant vanishes.

To order ϵ^3, η^0, the hexagonal solution was shown to be unstable since

$$\epsilon^2\sigma^{(2,0)} > 0.$$

To order $\epsilon^2\eta$, however, $\sigma = \sigma^{(0,0)} + \epsilon\eta\sigma^{(1,1)} + \epsilon^2\sigma^{(2,0)}$, it is found that $\epsilon\eta\sigma^{(1,1)}$ can be negative and of larger absolute magnitude than $\epsilon^2\sigma^{(2,0)}$, for a certain range of ϵ. Although to order ϵ^3, η^0, rolls were shown to be stable, they are now found to be unstable if there is an η effect. For non-hexagonal solutions $\sigma^{(1,1)}$ vanishes so there can be no suppressing of the instability predicted by a positive $\sigma^{(2,0)}$. The condition for the stability of hexagons is $\epsilon\eta R^{(1,1)} < 0$. Rolls are stable in the following amplitude range:

$$|\epsilon| \geqslant \frac{|2\eta R^{(1,1)} N_0|}{\sqrt{(3)}\,(L_\Delta - L_1)} \equiv |\epsilon_c|,$$

where
$$\frac{L_\Delta}{N_0^2} = 0{\cdot}18921 + \frac{0{\cdot}04566}{p_r} + \frac{0{\cdot}07089}{p_r^2},$$
$$L_1 = 0,$$

for two free boundaries; and for two rigid boundaries,

$$\frac{L_\Delta}{N_0^2} = 0{\cdot}19913 + \frac{0{\cdot}05116}{p_r} + \frac{0{\cdot}06509}{p_r^2},$$
$$\frac{L_1}{N_0^2} = \left(2{\cdot}46 - \frac{1{\cdot}57}{p_r} + \frac{2{\cdot}78}{p_r^2}\right) \times 10^{-3}.$$

Thus we arrive at a stability diagram just as Busse (1962) does in his study of varying physical properties.

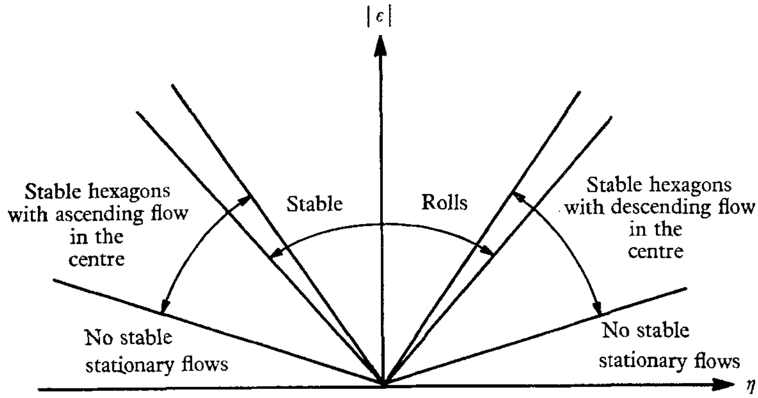


FIGURE 3. Stability diagram.

5. Conclusions

Within the framework of the Boussinesq equations, the degeneracy encountered in the linear theory has been removed by the non-linearities and the changing mean temperature. The static state is unstable to finite amplitude disturbances at Rayleigh numbers below the critical point predicted by linear theory. There is a range of Rayleigh number near the critical for which hexagonal flows are stable with the direction of flow determined by the sign of η. At higher Rayleigh number the roll is found to be the stable flow. A hysteresis in the heat flux curve is expected as the critical Rayleigh number is approached from below or from above. Here is an effect whose magnitude in a laboratory experiment can be altered at will simply by turning a dial. In part 2 (Krishnamurti 1968), some of these results are tested in a laboratory experiment.

This paper is condensed from a Ph.D. dissertation, University of California, at Los Angeles. I wish to express my deepest gratitude to Professor W. V. R. Malkus for suggesting this problem and for his continued interest and encouragement. I also wish to thank Dr Friederick Busse for many helpful discussions. The research was supported by the National Science Foundation under grant GP-2414.

REFERENCES

BÉNARD, H. 1900 *Revue General des Sciences Pures et Appliquées*, **11**, 1261, 1309.

BÉNARD, H. 1901 *Ann. Chem. Phys.* **23**, 62.

BUSSE, F. H. 1962 Dissertation, University of Munich. Translation from the German by S. H. Davis, The RAND Corporation, Santa Monica, California, 1966.

CHANDRASEKHAR, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford: Clarendon Press.

DEBLER, W. R. 1966 *J. Fluid Mech.* **24**, 165.

KRISHNAMURTI, R. 1967 Dissertation, University of California at Los Angeles.

KRISHNAMURTI, R. 1968 *J. Fluid Mech.* **33**, 457.

LORTZ, D. 1961 Dissertation, University of Munich.

MALKUS, W. V. R. 1964 Course lectures, Summer Study Program in Geophysical Fluid Dynamics, Woods Hole Oceanographic Institution.

MALKUS, W. V. R. & VERONIS, G. 1958 *J. Fluid Mech.* **4**, 225.

MIHALJAN, J. M. 1962 *Astrophys. J.* **136**, 1126.

PALM, E. 1960 *J. Fluid Mech.* **8**, 183.

PELLEW, A. & SOUTHWELL, R. V. 1940 *Proc. Roy. Soc.* A **176**, 312.

RAYLEIGH, LORD 1916 *Phil. Mag.* **32**, 529.

SCHLÜTER, A., LORTZ, D. & BUSSE, F. 1965 *J. Fluid Mech.* **23**, 129.

SEGEL, L. A. & STUART, J. T. 1962 *J. Fluid Mech.* **13**, 289.

SPARROW, E., GOLDSTEIN, R. J. & JONSSON, V. K. 1964 *J. Fluid Mech.* **18**, 513.

SPIEGEL, E. A. & VERONIS, G. 1960 *Astrophys. J.* **131**, 442.

TIPPELSKIRCH, H. 1956 *Beiträge zur Physik der Atmosphäre*, **29**, 37.